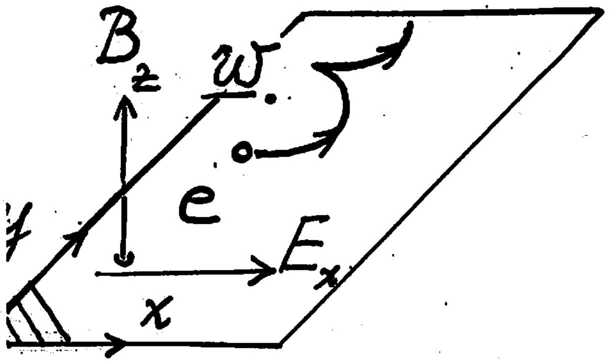
II. INTEGER QHE

von Klitzing '80.



$$eE_x = \frac{e}{c} v_y B_z$$

n : density

$$\Longrightarrow \quad j_y = env_y = \frac{ecn}{B_z} E_x$$

$$= \frac{e^2}{h} \cdot \underbrace{\frac{n}{B_z / \frac{h}{ec}}}_{\nu} \cdot E_x$$

QM: Landau spectrum

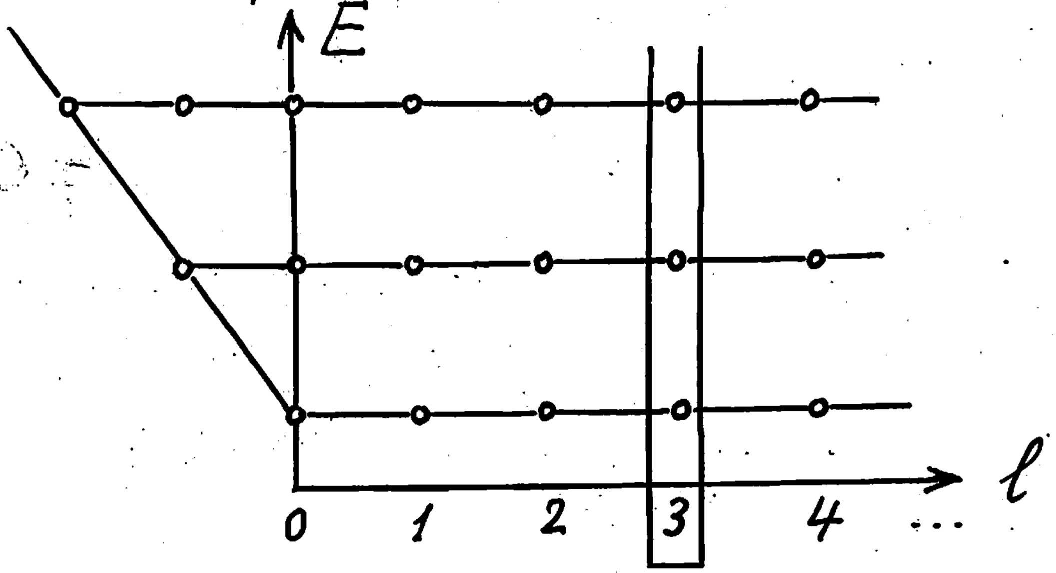
$$[\pi_x, \pi_y] = [w_y, w_x] = im\hbar\Omega_C$$

$$H = (2m)^{-1}[\pi_x^2 + \pi_y^2], \quad \Omega_C = eB^{(0)}/mc$$

$$\Longrightarrow \quad E_n = \hbar\Omega_C\left(n + \tfrac{1}{2}\right), \quad n = 0, 1, 2, \cdots$$

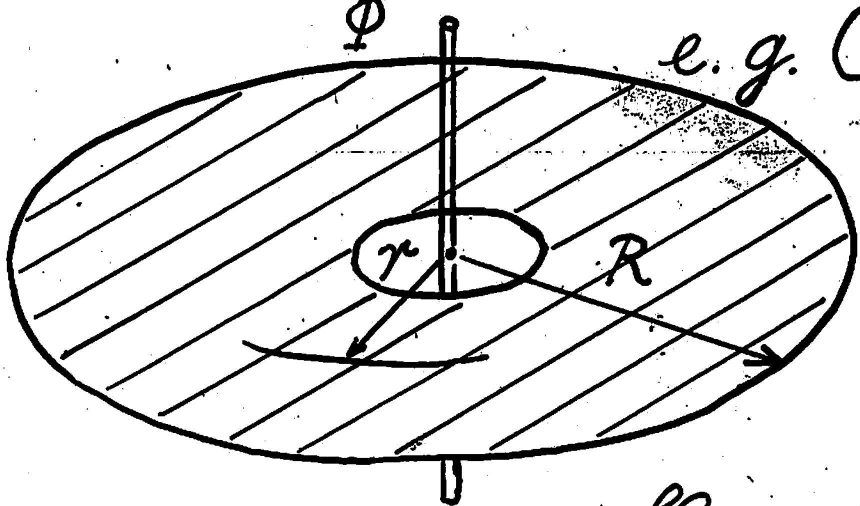
E_n is ∞ deg., (states labelled by n; $\ell = -n, -n+1, \cdots -1, 0, 1, 2, \cdots$)

w-plane = "phase plane"



Geometry of finite sample

e.g. Corbino



Φ : magnetic flux through h.

$\ell \sim m r^2 \Omega_c / \hbar$

Das Elektron in einem homogenen Magnetfeld $\vec{B}$.

Wählen $\vec{A} = \frac{1}{2}(\vec{B} \wedge \vec{x})$, so dass $\vec{\nabla} \wedge \vec{A} = \vec{B}$ und $\vec{\nabla} \cdot \vec{A} = 0$.

Ohne Verlust an Allgemeinheit können wir $\vec{B}$ in $(+z)$-Richtung wählen, d.h. $\vec{B} = (0,0,B)$. Wir definieren

$$\pi_x = \frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{e}{c}A_x = \frac{\hbar}{i}\frac{\partial}{\partial x} + \frac{e}{2c}yB,$$

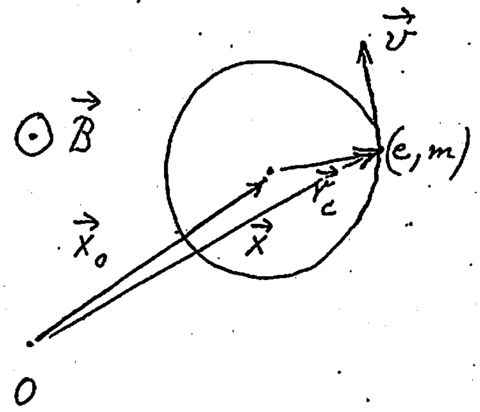
$$\pi_y = \frac{\hbar}{i}\frac{\partial}{\partial y} - \frac{e}{c}A_y = \frac{\hbar}{i}\frac{\partial}{\partial y} - \frac{e}{2c}xB,$$

$$\pi_z = \frac{\hbar}{i}\frac{\partial}{\partial z}. \tag{104}$$

$$H = \frac{1}{2m}\left(\pi_x^2 + \pi_y^2 + \pi_z^2\right) \otimes \mathbb{1} - \mathbb{1} \otimes \frac{g\mu_B}{2} B\sigma_3 \tag{105}$$

Die klassische Larmor- oder Zyklotronfrequenz ist

$$\omega_c = \frac{eB}{mc}. \tag{106}$$



$$\vec{x} = \vec{x}_0 + \vec{r}_c$$

$$\vec{v} = \dot{\vec{x}} = \dot{\vec{r}}_c \;,\; \text{da } \dot{\vec{x}}_0 = 0$$

$$\text{Zentrifugalkraft} = \frac{mv^2}{r_c}, \quad r_c = |\vec{r}_c|.$$

$$\text{Lorentzkraft} = eB\frac{v}{c}.$$

Also: $\quad \dfrac{mv^2}{r_c} = eB\dfrac{v}{c}$, oder $\quad \omega_c \equiv \dfrac{v}{r_c} = \dfrac{eB}{mc} \qquad (107)$

Weiter gilt klassisch:

$$\vec{x}_0 = \vec{x} - \vec{r}_c = \text{const.},$$

oder wegen (107),

$$\vec{x}_0 = \vec{x} + \varepsilon \frac{\vec{v}}{\omega_c} = \text{const.}, \qquad (108)$$

wo $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Da $m\vec{v} = \vec{\pi}$, folgt, dass auch

$$\vec{w} = \vec{\pi} - m\omega_c \varepsilon \vec{x} = \text{const.}, \qquad (109)$$

(Multiplikation von (108) mit $-m\omega_c\varepsilon$!).

In der <u>QM</u> werden aus π_x, π_y, π_z die Operatoren in (104). Durch <u>Nachrechnen</u> verifiziert man leicht, dass

$$[\pi_x, \pi_y] = i\frac{eB}{c}\hbar \mathbb{1} = i\hbar m\omega_c \mathbb{1}, \; [\pi_{x/y}, \pi_z] = 0, \qquad (110)$$

und $\quad [w_x, w_y] = -i\hbar m\omega_c \mathbb{1}, \; [w_i, \pi_j] = 0. \qquad (111)$

Daraus folgt problemlos, dass

$$[H, w_j] = 0, \quad [H, \pi_z] = 0, \quad [H, \pi_i] = i\hbar\omega_c \varepsilon_{ij} \pi_j \qquad (112)$$

Schliesslich verifiziert man leicht, dass

$$L_z = \frac{\varpi_x^2 + \varpi_y^2 - \pi_x^2 - \pi_y^2}{2m\omega_c} = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \tag{113}$$

d.h. L_z ist die z-Komponente des Bahndrehimpuls-operators. Aus (110), (111) und (112) folgt dann, dass

$$[L_z, \pi_i] = i\hbar\,\varepsilon_{ij}\,\pi_j\,, \quad [L_z, \varpi_i] = i\hbar\,\varepsilon_{ij}\,\varpi_j\,, \tag{114}$$

und

$$[H, L_z] = 0,$$

wie es sich gehört!

Die Operatoren π_x und π_y sind die Generatoren von Translationen in der $(x-y)$-Ebene. Die erste Gleichung in (110) entspricht den Heisenberg'schen Vertauschungsrelationen für p und x. Sie sagt daher, dass die Translationen in der $(x-y)$-Ebene <u>projektiv</u> dargestellt werden; (<u>„magnetische Translationen“</u>!).

Auf grund des von Neumann'schen Eindeutigkeitssatzes für irreduzible Darstellungen der Heisenberg'schen VR, können π_x und π_y wie folgt gewählt werden:

$$\pi_x = i\,\frac{eB}{c}\,\hbar\,\frac{\partial}{\partial \xi}\,, \quad \pi_y = \xi - p_y\,, \tag{115}$$

wo p_y eine beliebige reelle Zahl ist.

Nach (111) gilt Analoges für w_x, w_y:

$$w_x = \eta \quad , \quad w_y = i\,\frac{eB}{c}\,\hbar\,\frac{\partial}{\partial \eta} \qquad (116)$$

Nun setzen wir $\xi - p_y =: \frac{eB}{\sqrt{mc^2}}\,u$, $\frac{\partial}{\partial \xi} = \frac{\sqrt{mc^2}}{eB}\,\frac{\partial}{\partial u}$. (11

Dann ist $H = H_0 + H_1$, wo

$$H_0 = \left[-\frac{\hbar^2}{2}\,\frac{\partial^2}{\partial u^2} + \frac{\omega_c^{\,2}}{2}\,u^2 \right] \otimes \mathbb{1} \,, \qquad (118)$$

$$H_1 = \frac{\pi_z^{\,2}}{2m} \otimes \mathbb{1} - \mathbb{1} \otimes \frac{g\mu_B}{2}\,B\,\sigma_3 \,.$$

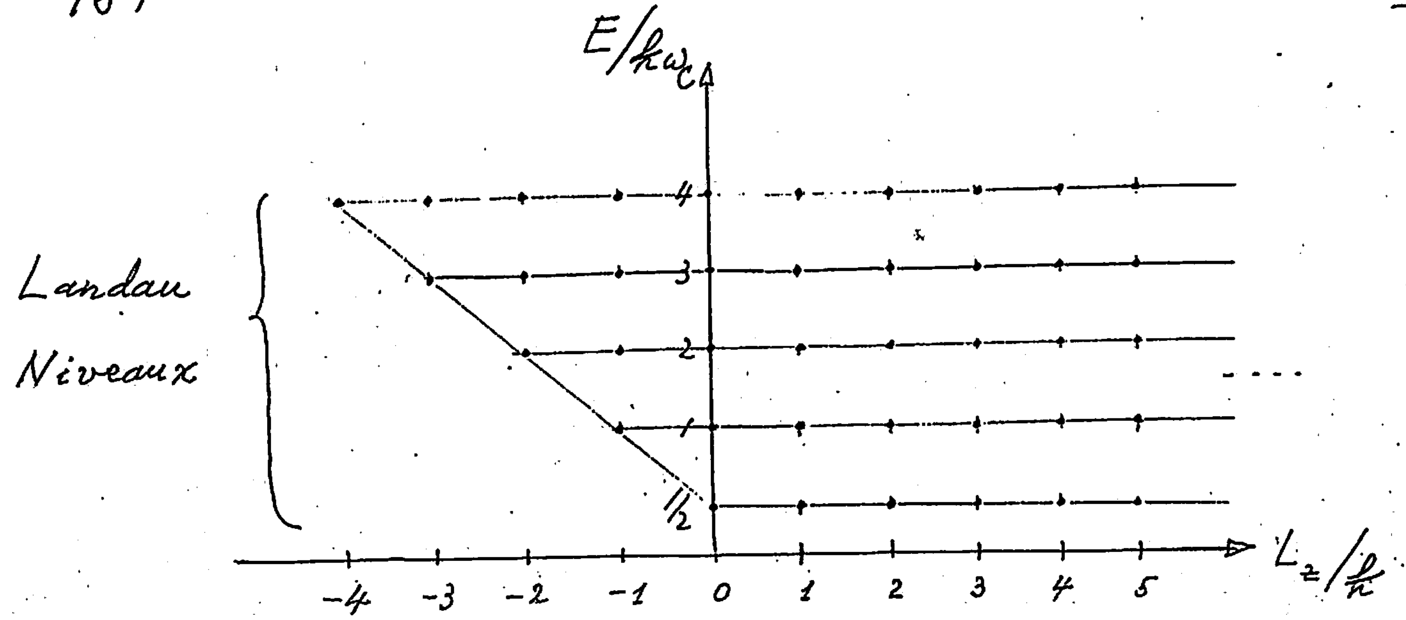
H_0 ist der Hamilton Operators eines <u>ein-dimensionalen harmonischen Oszillators.</u> Daher sind die Eigenwerte von H_0 die Energiewerte $E_n = \hbar\omega_c\,(n + \frac{1}{2})$, und die Eigenfunktionen sind Hermite Funktionen in u.

Mit den gleichen Argumenten wie für H_0 sieht man, dass das Spektrum von L_z aus der Menge

$$\{\hbar l = \hbar\,(m-n) :\; n, m = 0, 1, 2, \cdots \} \qquad (119)$$

besteht, d.h. das gemeinsame Spektrum von H_0 und L_z besteht aus

$$\sigma(L_z, H_0) = \left\{ \hbar\,\left(l, \omega_c (n + \tfrac{1}{2})\right) : n = 0, 1, 2, \cdots ,\; l = -n, -n+1, \ldots, 0, 1, \ldots \right\} \qquad (120)$$



Jedes Landau Niveau ist unendlich entartet. Die Entartung kann durch die z-Komponente des Drehimpulses indiziert werden. Im n^{ten} Landau Niveau sind die Eigenwerte von L_z die Zahlen $-\hbar n, -\hbar(n+1), \cdots, 0, \hbar, \cdots$.

Schliesslich bemerken wir, dass H_0 und H_1 vertauschen, und die zwei Terme in H_1 vertauschen auch. Damit ist das Spektrum von H durch

$$\hbar\omega_c\left(n+\frac{1}{2}\right) \pm \frac{g\mu_B}{2} B + \frac{(\hbar k)^2}{2m}, \qquad (121)$$

$n = 0, 1, 2, \cdots,$ $k \in \mathbb{R}$ gegeben. Daraus sehen wir, dass für $\mu_B = \dfrac{e\hbar}{2mc}$ und $g = 2$ ein Elektron im $(n-1)^{\text{ten}}$ Landau Niveau mit Spin "auf" die selbe Energie hat wie ein Elektron im n^{ten} Landau

Niveau mit Spin "ab". Diese Entartung ist die Folge einer quantenmechanischen Supersymmetrie. Für $g \neq 2$ ist diese aber gebrochen.

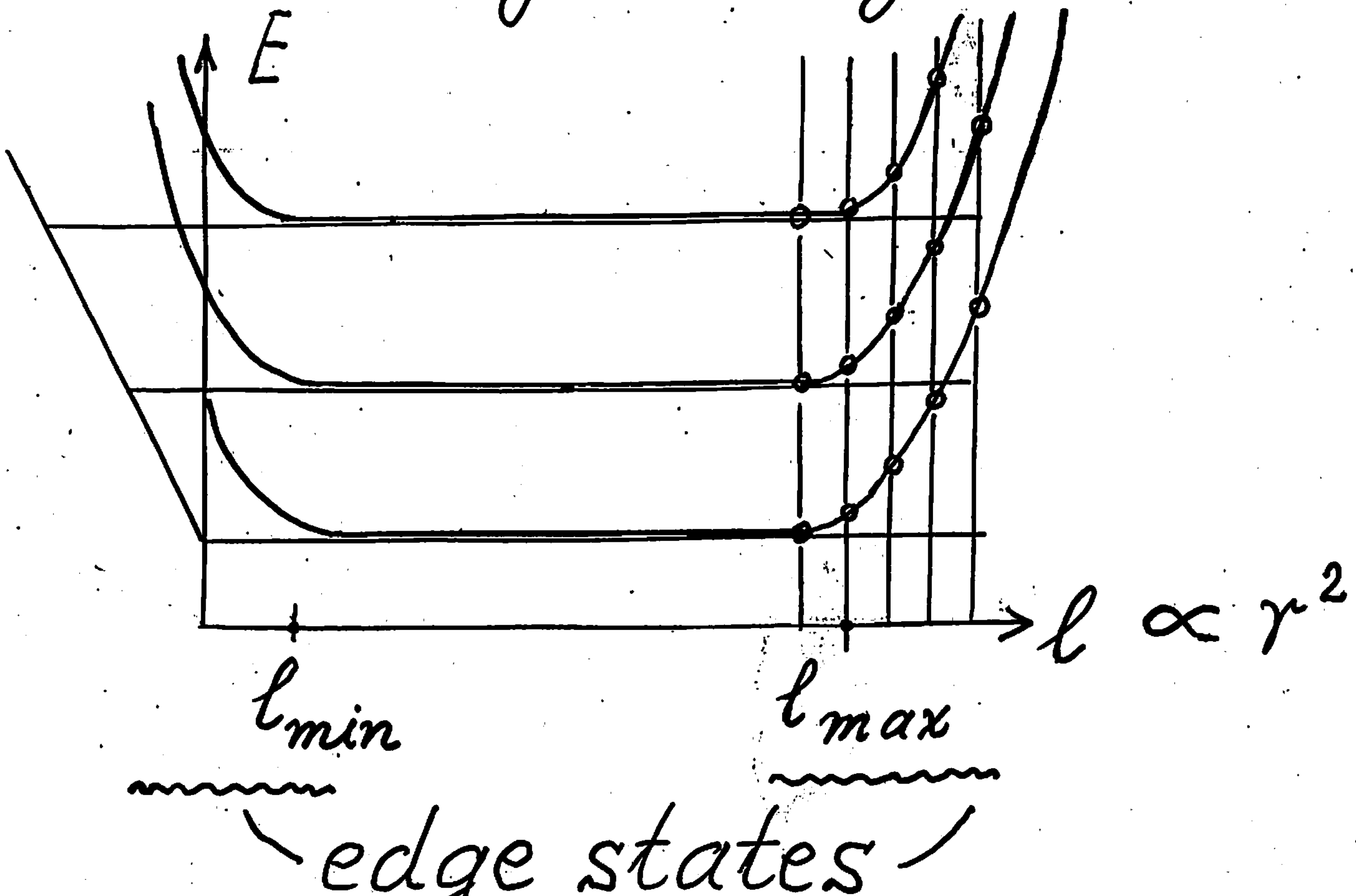
Die hier dargestellten Resultate stammen im Wesentlichen von L. D. Landau.

Edge potential, V, rot. inv.

$\Rightarrow$ V commutes with L_z,

$\Rightarrow$ ℓ is "good quantum #"

$\Rightarrow$ Effect of V on $E_{n,\ell}$ from non-deg., analytic P.T.

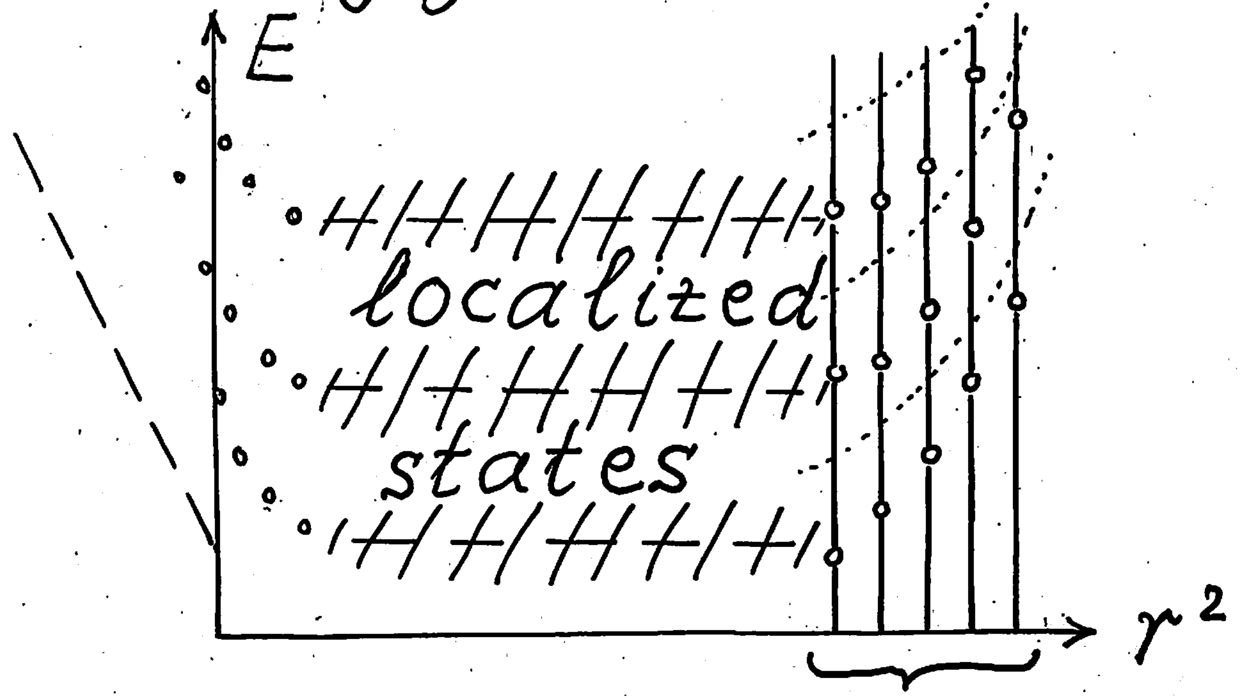


energies of edge states "non-degenerate"

#.

⇒ Effect of disorder pot., v, on spectrum of edge states "negligible", (F-G-W, M-M-P)

⇒ Energy spectrum, $v \neq 0$.

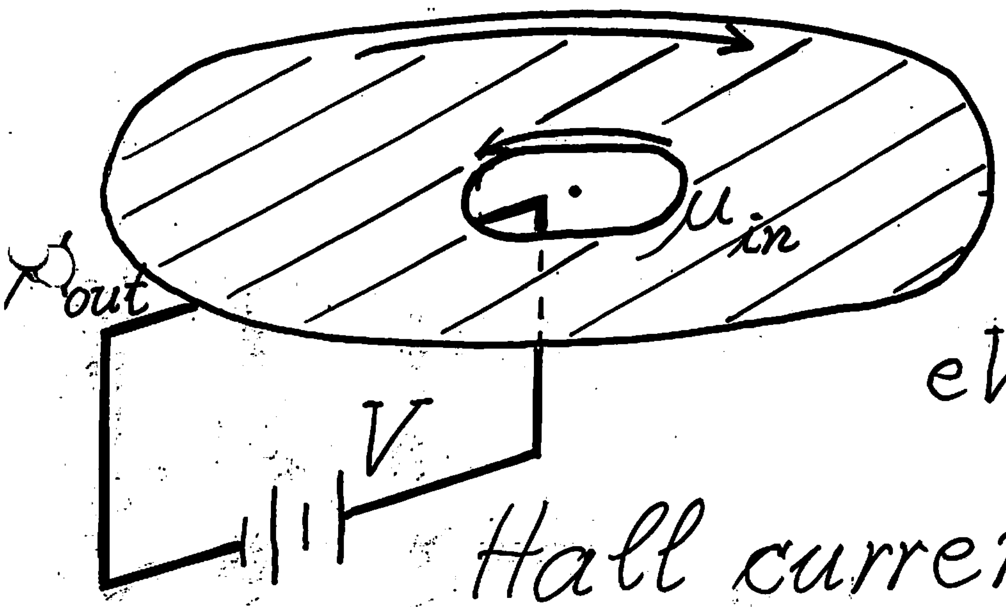


bulk localization: F-S, ...

ext. nature of edge states:

F-G-W, M-M-P; ("pos. comm.")

Turn on voltage drop in radial direction:



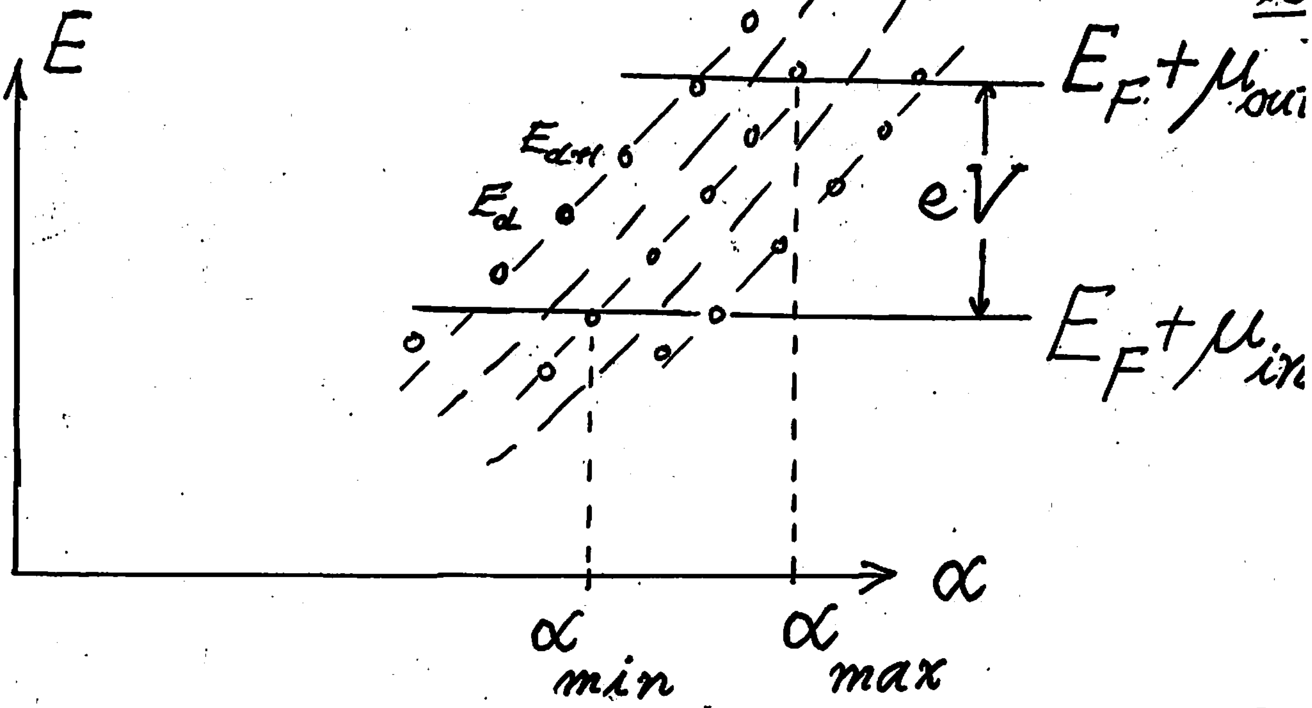
$$eV = \mu_{out} - \mu_{in}$$

Hall current, I_H, in azim. dir., <u>loc. near edges.</u>

$$G_H \equiv \frac{e^2}{h} \sigma_H = \frac{I_H}{V}$$

Fermi energy at outer edge: $E_F + \mu_{out}$,

at inner edge: $E_F + \mu_{in}$



Consider single edge channel. Current, I_α, carried by edge state α.

$$I_\alpha = \frac{1}{c}\left.\frac{\partial E_\alpha}{\partial \Phi}\right|_{\Phi=0}$$

$$\simeq \frac{e}{h}\int_0^{h/ec} d\Phi \; \frac{\partial E_\alpha(\Phi)}{\partial \Phi}$$

$$= \frac{e}{h}\left(E_{\alpha+1} - E_\alpha\right) \qquad *$$

Hall current per channel, ς.

$$I_H^{\varsigma} = \sum_{\alpha_{min} \leqslant \alpha \leqslant \alpha_{max}} I_\alpha$$

$$\simeq \frac{e}{h}\left(E_{\alpha_{max}} - E_{\alpha_{min}}\right)$$

$$= \frac{e^2}{h} V$$

$$I_H = \sum_{\text{filled } \varsigma} I_H^{\varsigma} \equiv G_H V$$

$$\Rightarrow G_H = \frac{e^2}{h} \times \#\text{filled edge channels}$$

* from

$$\frac{C}{R} \geqslant \frac{\partial E_\alpha(\Phi)}{\partial \Phi} \geqslant \frac{C'}{R}$$

(F-G-W)

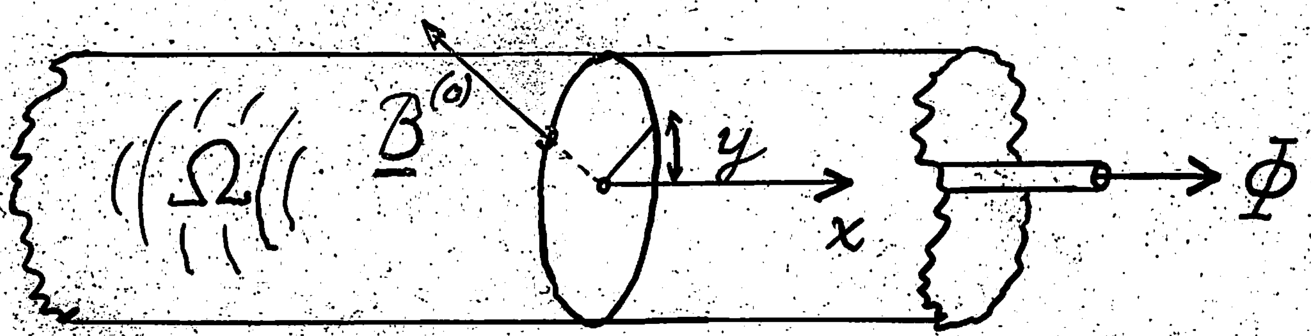
Plateaux of σ_H as fu. of filling factor ν: Because of <u>localization</u> of bulk states, # filled edge channels loc. const. in ν!

Result insensitive to sample geometry, temp. T (small). Is a pure "equilibrium" result.

3. Microscopic Theory

3.1. Non-int. electrons - IQHE

Electron gas on ∞ cylinder of radius $L/_{2\pi}$



$$m_{el} = e = \hbar = 1; \quad |\underline{B}^{(0)}| = B = \Omega_c$$

$$p_x = -i\frac{\partial}{\partial x}, \quad p_y = -i\frac{\partial}{\partial y},$$

$$0 \leq y < L$$

$$H_0(\Phi) = \frac{1}{2}p_x^2 + \frac{1}{2}\left(p_y - Bx + \frac{\Phi}{L}\right)^2$$

E.v. problem solved by

separation of variables:

$$\psi(x,y) = e^{i\frac{2\pi\ell}{L}y} h(x), \quad \ell \in \mathbb{Z}$$

$$(H_0\psi)(x,y) = e^{i\frac{2\pi\ell}{L}y}\Big[-\frac{1}{2}h''(x) + \frac{1}{2}\Big(\underbrace{\frac{2\pi\ell}{L} + \frac{\Phi}{L}}_{\equiv Bx_{\ell,\Phi}} - Bx\Big)^2 h(x)\Big]$$

$\Rightarrow \operatorname{spec} H_0(\Phi) = \left\{E_n = B\left(n + \frac{1}{2}\right)\right\}_{n=0}^{\infty}$

E_n ∞ deg., eigenstates labelled by (n,ℓ) concentrated near $x_{\ell,\Phi} = \frac{2\pi\ell + \Phi}{BL}$

$\Rightarrow$ Periodicity in Φ, period 2π.

$$H(\Phi) := H_0(\Phi) + v_\omega(x,y) + W(x)$$

v_ω : random potential

$$|v_\omega(x,y)| \leq \kappa \ll B$$

$W(x)$: edge potential

$$W(x) = 0,\ x \leq 0;\ W(x) \sim x^\alpha,$$

$$\alpha \geq 2,\ x \geq 0.$$

$$H_{edge}(\Phi) := H_0(\Phi) + W(x),$$

$$H_{bulk}(\Phi) := H_0(\Phi) + v_\omega(x,y).$$

spec$(H_{bulk}(\Phi))$ has gaps

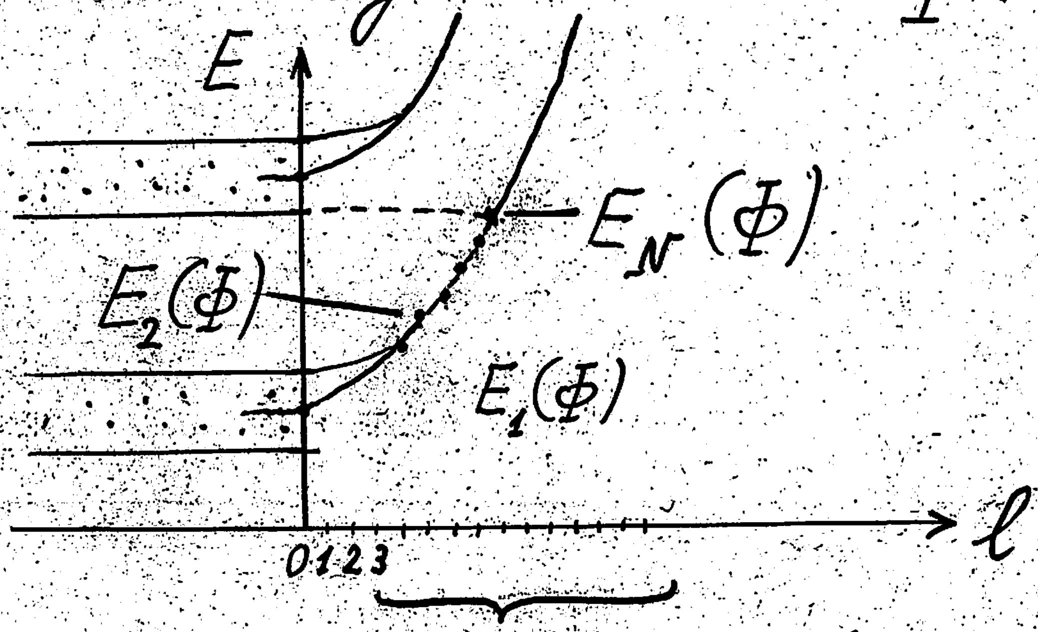
$$G_n = \left[(n+\tfrac{1}{2})B + \kappa,\ (n+\tfrac{3}{2})B - \kappa\right]$$

$G_n^{(\varepsilon)}$: G_n shrunk by ε.

$$\operatorname{spec}(H(\Phi)) \cap G_n^{(\varepsilon)} = \{E_k(\Phi)\}_{k=1}^{N}$$

$$E_1(\Phi) \leqslant \cdots \leqslant E_N(\Phi),$$

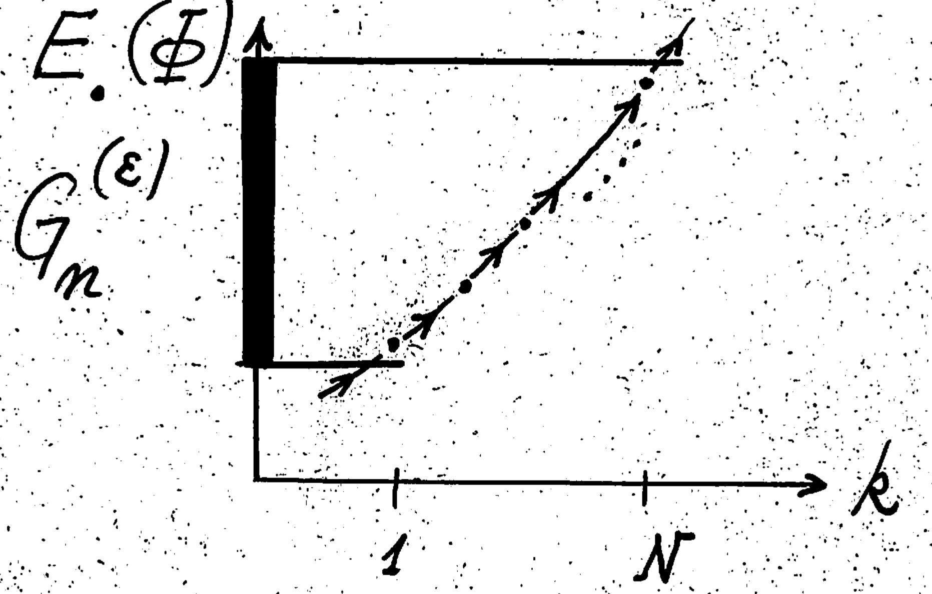
analytic in Φ.



edge states $\leftrightarrow x_{\ell,\Phi} > 0$

In gap $G_n^{(\varepsilon)}$, $E_k(\Phi)$ computable from $\operatorname{spec}(H_{\mathrm{edge}}(\Phi))$ by analytic perturbation theory in v_ω; anal. in Φ.

Spectral flow:



(1) $$\beta_1 \leq L \frac{d}{d\Phi} E_k(\Phi) \leq \beta_2$$

(2) $$\frac{d}{d\Phi} E_k(\Phi) = j_k(\Phi) =$$

$$= \frac{1}{L} \left\langle \Psi_k(\Phi), \left(p_y - Bx + \frac{\Phi}{L}\right) \Psi_k(\Phi) \right\rangle$$

current carried by $\Psi_k(\Phi)$.

(3) $$E_k(\Phi + 2\pi) = E_{k+1}(\Phi)$$

$$\Rightarrow \quad \frac{2\pi \beta_1}{l} \leq \left| E_{k+1}(\Phi) - E_k(\Phi) \right| \leq \frac{2\pi \beta_2}{l}$$

Determination of $(\sigma_H)_{edge}$

Let $\Delta := [E_F - \frac{V}{2}, E_F + \frac{V}{2}] \subset G_n^{(\varepsilon)}$, $(\varepsilon > 2\pi\beta_2/L)$. Consider one branch of edge states (out of $n+1$ = # filled Landau bands). Then

$$(\sigma_H)_{edge}^{1\,branch} = \frac{1}{V} \sum_{E_k(0)\in\Delta} j_k(0)$$

$$\overset{(2)}{\approx} \frac{1}{V} \int_0^{2\pi} \frac{d\Phi}{2\pi} \sum_{E_k(\Phi)\in\Delta} \frac{dE_k(\Phi)}{d\Phi}$$

$$\overset{(3)}{=} \frac{1}{2\pi V} \sum_{k=k_{min}}^{k_{max}} \left(E_{k+1}(0) - E_k(0)\right)$$

$$= \frac{1}{2\pi V}\left(E_{k_{max}+1}(0) - E_{k_{min}}(0)\right)$$

$$\approx \frac{1}{2\pi}\left(\times\,\frac{e^2}{\hbar}\right) \qquad (4)$$

$\Rightarrow$ In the limit where $L \to \infty$:

$$(\sigma_H)_{edge} = n+1, \qquad (5)$$

in units, where $\frac{e^2}{h} = 1$,

provided $E_F \in G_n^{(\varepsilon)}$!

Integer quantization of $(\sigma_H)_{edge}$!

Can we understand why

$$(\sigma_H)_{edge} = (\sigma_H)_{bulk} \; ?$$

$(\sigma_H)_{edge}$ from flow of edge spect. of syst. on half-∞ cylinder of radius $L/2\pi$, (or on plane with disk of radius $L/2\pi$ cut out), in limit $L \to \infty$.

$(\sigma_H)_{bulk}$ from "spectral flow" of system in "punctured plane" via "index of pair of projections" (Bellissard; Avron, Seiler, Simon):

$$(\sigma_H)_{bulk} = tr\left(P_F(2\pi) - P_F\right)^{2N+1},$$

$$N \geq 1.$$

Here P_F is proj. onto eigen-states of energies $\leq E_F \in G_n^{(\varepsilon)}$,

$$P_F(2\pi) := U(\Phi = 2\pi)\, P_F\, U(\Phi = 2\pi)^*,$$

where

$$U(2\pi) := \exp(i \arg \underline{x}).$$

$U(2\pi)$ turns on a magn. flux $\Phi = 2\pi$ through $\underline{x} = 0$.

<u>Indices of projections:</u>

Let P, Q be 2 proj. s.t. $(P-Q)^M$ is trace-class, for M large enough. Let N be s.t. $2N+1 \geq M$. Then

$$\mathrm{Ind}(P,Q) := \mathrm{tr}(P-Q)^{2N+1}$$

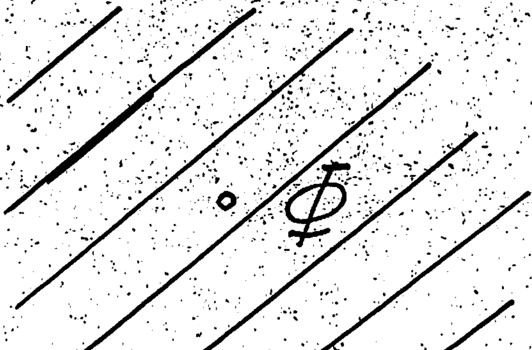
$$= -\,\mathrm{Ind}(Q,P) \quad (6)$$

$$= \mathrm{Ind}(UPU^*, UQU^*), \quad (7)$$

for arb. unitary op U.

Let (P,Q), (Q,R) each have finite index, $P-Q$ or $Q-R$ <u>compact</u>. Then

$$\mathrm{Ind}(P,R) = \mathrm{Ind}(P,Q) + \mathrm{Ind}(Q,R) \quad (8)$$



$P_F(\Phi)$ $\quad$ $P_F^L(\Phi)$

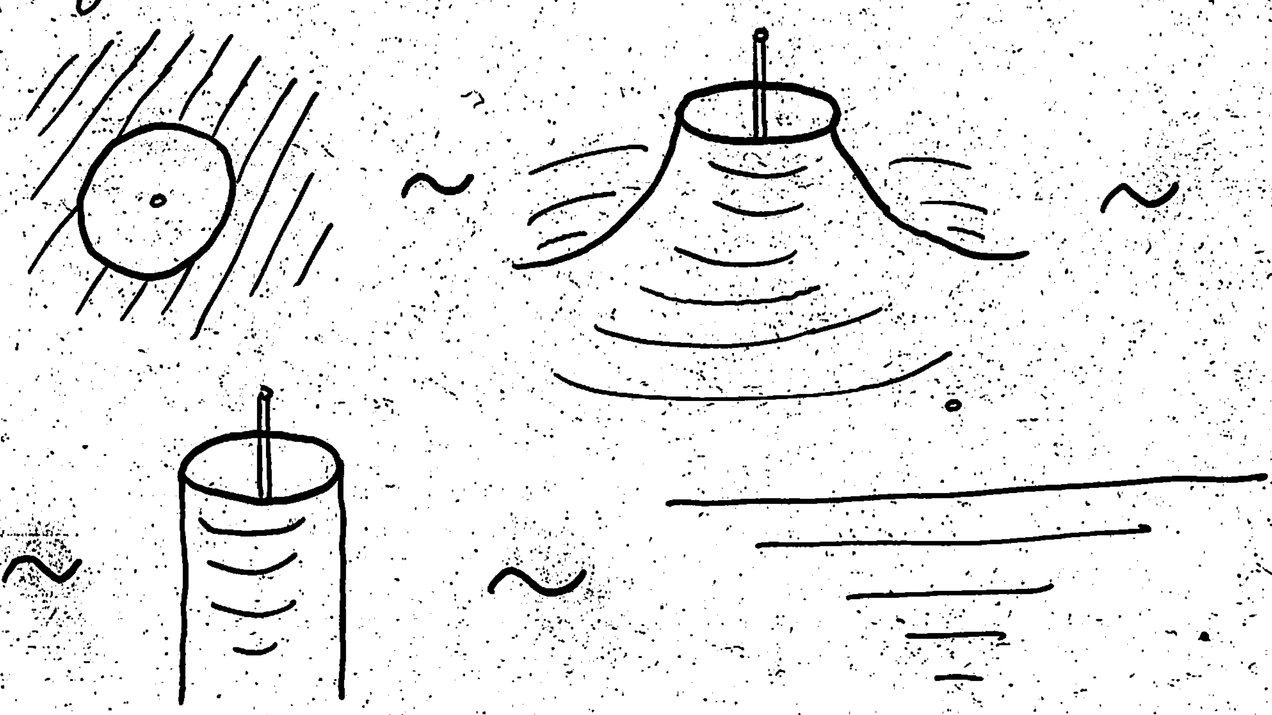
Then

$$
\begin{aligned}
(\sigma_H)_{bulk} &\overset{(6)}{=} \mathrm{Ind}\left(P_F(2\pi), P_F\right) \\
&\overset{(8)}{=} \mathrm{Ind}\left(P_F(2\pi), P_F^{L}(2\pi)\right) && \mathrm{I} \\
&\quad + \mathrm{Ind}\left(P_F^{L}(2\pi), P_F^{L}(0)\right) && \mathrm{II} \\
&\quad + \mathrm{Ind}\left(P_F^{L}(0), P_F\right) && \mathrm{III}
\end{aligned}
$$

$\mathrm{I} + \mathrm{III} = 0$, by (7) & (6).

<u>Claim:</u> $\mathrm{II} = (\sigma_H)_{edge}$

By (8),



Remark. Instead of syst. on half-∞ cyl., consider one on plane with disk of radius $\frac{L}{2\pi}$ cut out, $L \to \infty$, in order to define $(\sigma_H)_{edge}$

Credit for material on edge states, $(\sigma_H)_{edge} = (\sigma_H)_{bulk}$, etc.: J.F. (many lectures on QHE); Macris, Martin, Pulé; J.F, Graf, Walcher; De Bièvre, Pulé; Kellendonk, Richter, Schulz-Baldes, Elbau, Graf; Macris.

Mean-field theory (FQHE)

Consider N spin-polarized electrons in domain $\Omega \subset \mathbb{R}^2$ in uniform magn. field $\vec{B} \perp \Omega$.

$\sim$ N scalar fermions conf. to Ω, charge $-e$, magn. f. $\vec{B}$

$$\vec{B} = \text{curl}\, \vec{A}, \quad \vec{A} = \frac{B}{2}(-y, x, 0)$$

Two-body interactions

$$V(\underline{x}) = \frac{e^2}{|\underline{x}|} \quad (\times \text{ screening})$$

To study this system, may attach magn. flux

$$\Phi = 2\ell \frac{h}{e}, \quad \ell = 0, 1, 2, \cdots \quad (\text{Jain})$$

to each fermion. Since $2l$ is even, Aharonov-Bohm phases for exchange of 2 particles, or roundtrips of 1 particle, are $= 1$. $\Rightarrow$ Particles remain <u>fermions</u>; flux attachment described by unitary conjugation. Dynamics for different values of l are <u>gauge-equivalent</u>; properties of system independent of l!

Describe system in formalism of 2^{nd} quantization, using Berezin funct. int.: Action funct. for fermions with $2l$ units of magn. flux attached, density $n = \frac{N}{|\Omega|}$, given by:

$$S(\psi^*, \psi; A+\alpha)$$
$$= \int dt \int_\Omega d^2x \Big\{ \psi^*(i\partial_t - \alpha_0)\psi$$
$$- \frac{1}{2m}\left[(i\underline{\nabla} - \underline{A} + \underline{\alpha})\psi\right]^* \left[(i\underline{\nabla} - \underline{A} + \underline{\alpha})\psi\right]$$
$$- (\psi^*\psi - n) * (V \otimes \delta)(\psi^*\psi - n)$$
$$+ \frac{1}{2l}\left(\alpha_0 \,\mathrm{curl}\,\underline{\alpha} - \varepsilon^{ij}\alpha_i \partial_0 \alpha_j\right) \Big\}$$
$$+ \text{bd. terms} \qquad \text{CS-t.}$$

Eqs. of motion:

Variation w.r.to α_0 yields

$$\operatorname{curl}\underline{\alpha} = 2\ell\,\psi^*\psi \qquad (9)$$

Set

$$\underline{a} := -\underline{A} + \underline{\alpha}, \qquad b := \operatorname{curl}\underline{a} \qquad (10)$$

Then

$$\operatorname{curl}\underline{a} = 2\ell\,\psi^*\psi - \underbrace{\operatorname{curl}\underline{A}}_{B}$$

$$= 2\ell\left(\psi^*\psi - B/2\ell\right)$$

$$= 2\ell\left(\psi^*\psi - \frac{n}{2\ell\nu}\right), \qquad (11)$$

where $\nu = n/B$ is the filling factor of fluid.

In funct. int., (11) imposed after integration over α_0.

Hamiltonian corresp. to action S given by

$$H = \int_{\Omega} d^2x \left\{ \frac{1}{2m} \left[(i\underline{\nabla} + \underline{a})\psi \right]^* \left[(i\underline{\nabla} + \underline{a})\psi \right] + (\psi^*\psi - n) * V(\psi^*\psi - n) \right\}, \tag{12}$$

with constraint (11) imposed, and $[a_i, a_j] \propto \varepsilon_{ij}\, \delta$.

(i) $\nu = \frac{1}{2\ell}$ $\left(= \frac{1}{2}, \frac{1}{4}\right)$. Then (11) yields $\text{curl}\, \underline{a} = 2\ell(\psi^*\psi - n)$, (13)

hence

$$H = \int_{\Omega} d^2x \left\{ \frac{1}{2m} \left[(i\underline{\nabla} + \underline{a})\psi \right]^* \left[(i\underline{\nabla} + \underline{a})\psi \right] + \beta_1 (\psi^*\psi - n) * V(\psi^*\psi - n) + \beta_2\, \text{curl}\, \underline{a} * V\, \text{curl}\, \underline{a} \right\}$$

with constraint (13).

For V of pos. type, term

$$\int_\Omega \beta_2 \operatorname{curl} \underline{a} * V \operatorname{curl} \underline{a}, \quad \beta_2 > 0,$$

suppresses large curl $\underline{a}$'s.

$\Rightarrow$ (13) "automatically" satisfied $\rightarrow$ neglect it!

Then eliminate $\underline{a}$ by using eqs. of motion (or by funct. integration)

$\Rightarrow$ 2D system of fermions with <u>current-current int.</u>

For $V(\underline{x}) \propto |\underline{x}|^{-1-\gamma}$, $0 \leq \gamma \leq 1$, expect 2D Luttinger liquid!

(ii) $\nu = \dfrac{N}{2lN+1}$, $N = 1, 2, 3, \dots,$

constraint (11) $\Longrightarrow$

$$\mathrm{curl}\,\underline{a} = 2l\,(\psi^*\psi - n) - \frac{n}{N}$$

Introducing a chemical potential $\mu = \mu(n)$ s.t. $\langle \psi^*\psi \rangle - n = 0$, we find

$$b = \mathrm{curl}\,\underline{a} \approx -\frac{n}{N} \tag{14}$$

$$\to \quad \nu_{eff} = -\frac{n}{b} \approx N$$

$$H \approx \int_\Omega d^2x \left\{ \frac{1}{2m} \left[(i\underline{\nabla} + \underline{a}^{(0)})\psi \right]^* \times \left[(i\underline{\nabla} + \underline{a}^{(0)})\psi \right] - \mu(n)\,\psi^*\psi \right\}, \tag{15}$$

with $\mathrm{curl}\,\underline{a}^{(0)} = -\dfrac{n}{N}$

(15) describes 2D fluid with exactly N filled Landau levels $\Rightarrow R_L = 0$, thanks to gap in bulk spectrum.

This is Jain's "theory" of composite fermions.

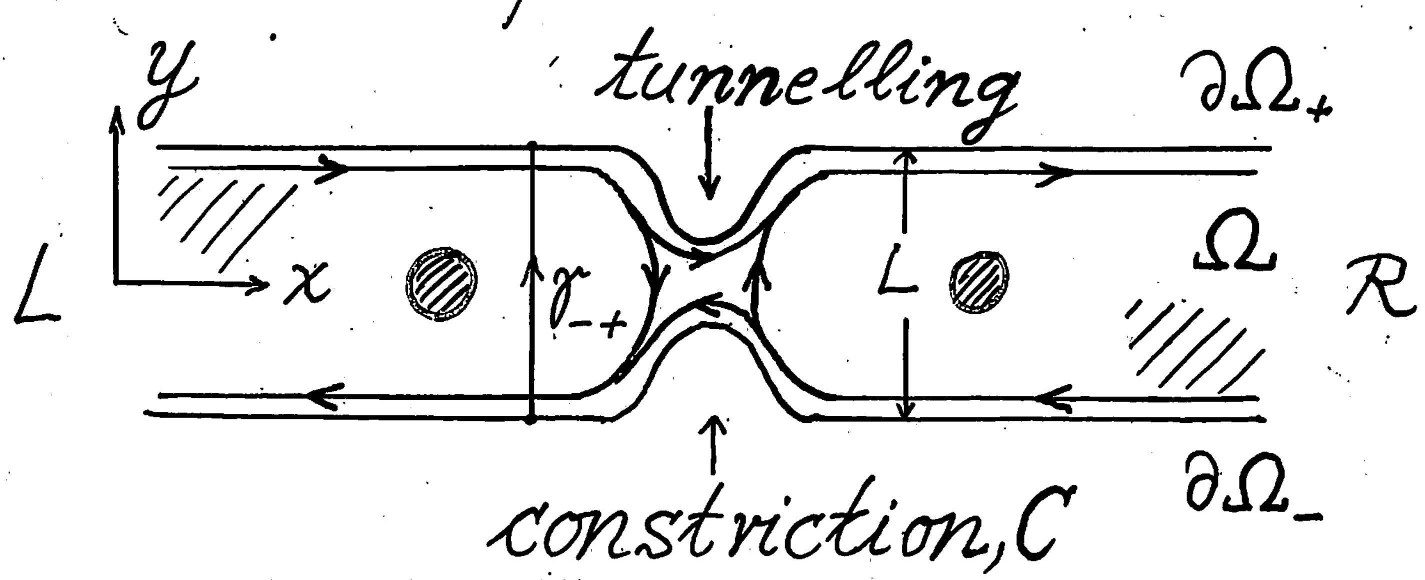
It appears to be somewhat accidental that this theory yields

$$\sigma_H = \nu = \frac{N}{2\ell N + 1}$$

Details are wrong!

5. TUNNELLING BETWEEN TWO OPPOSITE QH EDGES

- Suggest experiments by which fract. el. charges of quasi-particles can be measured: Tunnelling betw. QH edges through constriction, w. Aharonov-Bohm fluxes.



Effective action before constriction is introduced:

$$S^{eff}(A) = \frac{\sigma_H}{2} \int_{\Omega \times \mathbb{R}} A \wedge dA + \Gamma_{\partial\Omega}(a),$$

$$a = A\big|_{\partial\Omega}.$$

Imagine that E_1 only depends on t, x, not on y.

$\to$ Choose $A = (A_0, A_1, A_2)$, with A_0, A_1 <u>indep.</u> of y.

Define

$$\varphi(x,t) := \int_{\gamma_{-+}(x)} A_2(x,y,t)\, dy$$

"axion"

Then, with $\gamma_{-+} = \gamma_{-+}(x)$,

$$\dot{\varphi}(x,t) = \int_{\gamma_{-+}} \dot{A}_2(x,y,t)\,dy$$

$$= \int_{\gamma_{-+}} E_2(x,y,t)\,dy$$

$$= V(x,t) \quad \text{(voltage drop)}$$

$$\varphi'(x,t) = \int_{\gamma_{-+}} (\partial_x A_2)(x,y,t)\,dy$$

$$= \int_{\gamma_{-+}} B(x,y,t)\,dy$$

Eff. action is (CS action)

$$S^{eff}(A;\varphi) = \sigma_H \int dt \int dx\, \varphi \cdot E$$

$$+ W(a),$$

$$E := E_1, \quad a = A\big|_{\partial\Omega_+} = A\big|_{\partial\Omega_-}$$

$$W(a) = \underbrace{\Gamma_{\partial\Omega_+}(a) + \Gamma_{\partial\Omega_-}(a)}$$

gauge-invariant!

$$\propto \sigma_H \int dt \int dx \, (a^T)^2$$

$$j^\mu(x,t) = \delta S^{eff} \big/ \delta A_\mu(x,t) \, , \; \mu = 0,1$$

$$= -\sigma_H \, \varepsilon^{\mu\nu} \partial_\nu \varphi(x,t)$$

+ bd. term vanishing when $E = 0$

$$\Rightarrow \quad \delta Q_{L\to R} = \int dt \, j^1(x,t)$$

$$= \sigma_H \int dt \, \dot\varphi(x,t)$$

$$= \sigma_H \, \delta\varphi \qquad \#$$

$\delta\varphi(x) = \int V(x,t) \, dt$: voltage pulse

is equ. for an "ideal" quantum pump.

<u>Effect of tunnelling thru constriction:</u> Tunnelling term in action funct. is

$$\int dt \int dx \Big[\sum_\alpha t_\alpha(x)\, \bar\psi_{+\alpha}(x,t) \times \underbrace{\exp\Big(2\pi i q_\alpha \int_{\gamma_{-+}(x)} A_2(x,y,t)\, dy\Big)}_{\exp(2\pi i q_\alpha \varphi(x,t));} \psi_{-\alpha}(x,t) + h.c.\,(+ \leftrightarrow -)\Big]$$

saves gauge-invariance!

$\overset{(-)}{\psi}_{\pm\alpha}$: fields describing

α^{th} species of $\binom{\text{right}}{\text{left}}$ - moving modes of el. charge q_α. (These are vertex ops. of "chiral Luttinger liquids"; tunelling term $\underset{\text{bosonized}}{\propto}$ $:\cos(\chi_\alpha + 2\pi i q_\alpha \varphi):$)

Integrate out edge degrees of freedom → contribution $W_T(a;\varphi)$ to effective action, w. props.:

$$W_T(a;\varphi+\varphi_0) = W_T(a;\varphi);$$

$$\varphi_0 = q_{min}^{-1} = \underset{\substack{\Uparrow \\ N}}{l}\, \underset{\substack{\uparrow \\ \text{Hall den.}}}{d_H}.$$

$$\Longrightarrow\ j^\mu(x,t) = -\sigma_H \varepsilon^{\mu\nu}\partial_\nu \varphi(x,t) + \delta W_T(a;\varphi)\big/\delta a_\mu(x,t)$$

Charge transport from L to R corresp. to pulse $\delta\varphi$:

$$\delta Q = \sigma_H\, \delta\varphi + \int dt\, \frac{\delta W_T(a;\varphi)}{\delta a_1(x,t)}$$

<u>Lemma.</u>

2^{nd} term on R.S. of (4)
vanishes in an adiabatic pumping process, provided

$$\delta\varphi = n\varphi_0, \quad n \in \mathbb{Z}.$$

<u>Pf.</u> Gauge invariance $\Rightarrow$ current conservation $\Rightarrow$

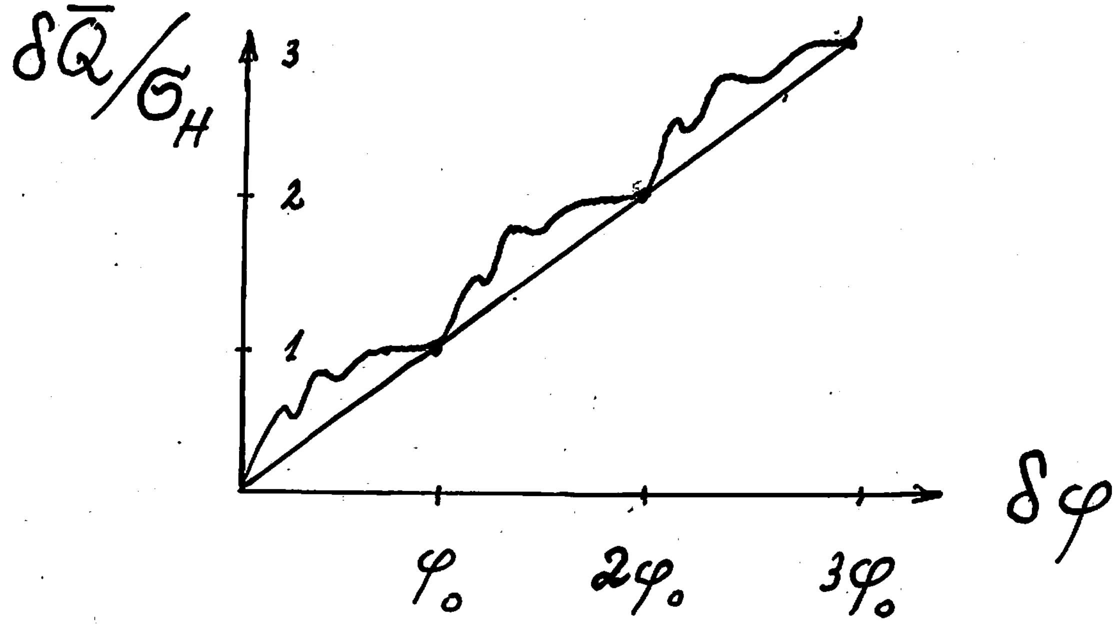
$$\partial_t \frac{\delta W_T(a;\varphi)}{\delta a_0(x,t)} + \partial_x \frac{\delta W_T(a;\varphi)}{\delta a_1(x,t)} = 0$$

$$\Rightarrow \quad \left.\frac{\delta W_T(a;\varphi)}{\delta a_1(x,t)}\right|_{a=0} = \partial_t U(\varphi;x,t)$$

$$U(\varphi;x,t) = -\int_{-\infty}^{x} dy \; \frac{\delta W_T(0;\varphi)}{\delta a_0(y,t)}$$

$U(\varphi; x, t)$ periodic in φ with period φ_0 ...

Q.E.D.



→ Determination of period φ_0, hence of q_{min}!

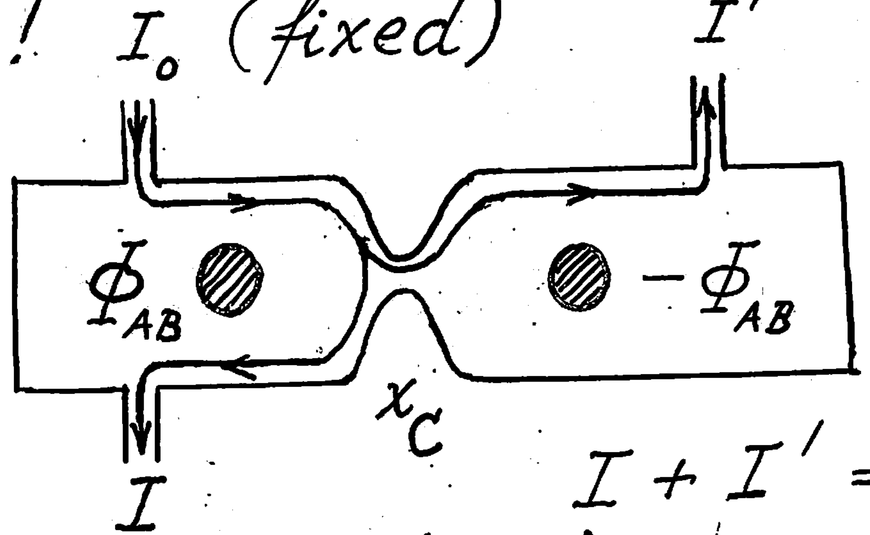
Dependence of (tunnelling) currents on Aharonov-Bohm fluxes:

$$\varphi'(x,t) = \int_{\gamma_+(x)} B(x,y,t)\,dy$$

$$\varphi(-\infty, t) = \varphi(+\infty, t) = 0$$

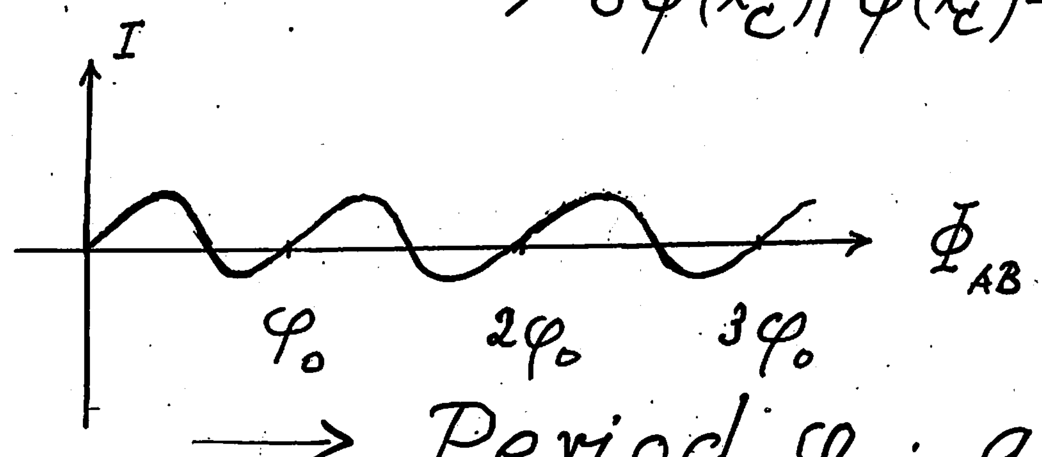
$$\Rightarrow \varphi(x, t) = \Phi_{AB}, \; x \text{ near } C$$

Edge- & tunnelling currents periodic in φ, hence in Φ_{AB} with period φ_0!



$$I + I' = I_0$$

$$I = \left. \delta W_T(0;\varphi) \middle/ \delta\varphi(x_C) \right|_{\varphi(x_C) = \Phi_{AB}}$$



$\longrightarrow$ Period φ_0; q_{min}!

Microscopic theory:

Edge states described by chiral CFT, with chiral algebra

$$\mathcal{C} \otimes \widehat{u}(1)$$

↑

chiral ($\hat{u}(1)$-) current algebra gen. by chiral el. edge current

$$j = \sqrt{\sigma_H}\, \partial_{\pm} \chi$$

Tunnelling term

$$\propto \,:\cos r(\chi + \varphi):\,(x_c)\,\cdots$$

Boundary CFT! (B-F)

HIGHER-DIM. COUSINS OF QHE – w. APPLICATIONS

J. Fröhlich

ETHZ - IHÉS

with Alekseev, Cheianov, Pedrini, Werner

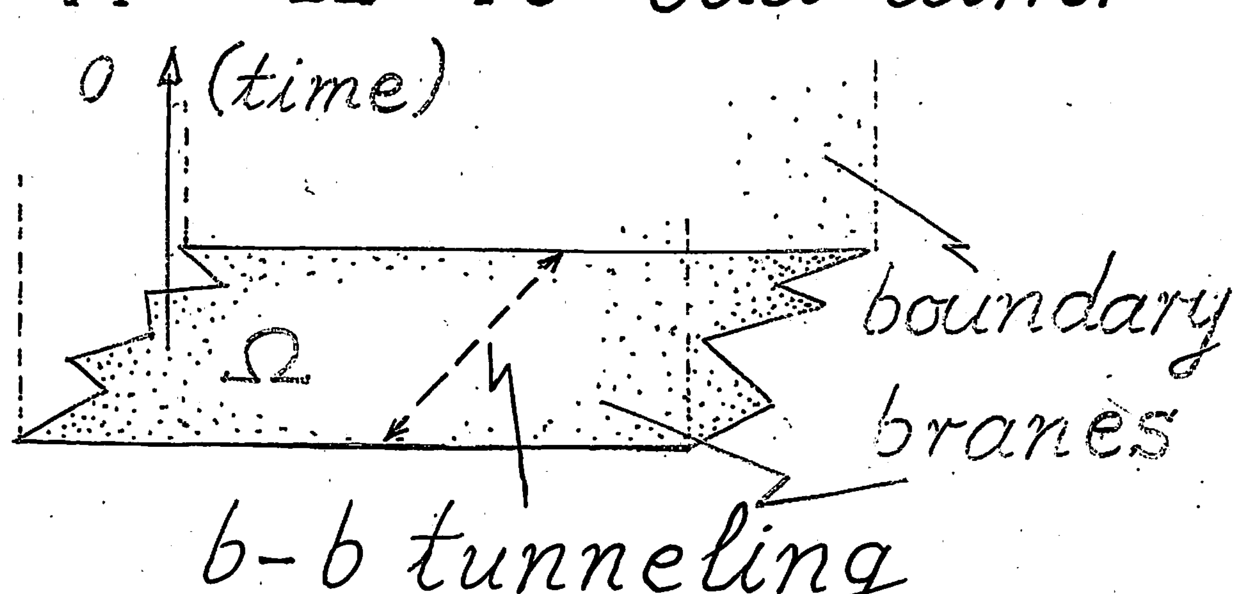
Exercise in anomaly cancellation & extra dimensions

Our starting point:

Massive matter in slab, Ω of 2, 4, ⋯ dim. space, e.g. 2D e^- gas in ext. magn. field (incompr.); or $2n$ D relat. matter, incl. 2^n comp. charged Dirac fermions, with $M = M_{GUT}, M_P$

$\Rightarrow P, T$ broken!

$$\Lambda = \Omega \times \mathbb{R} \quad \text{odd dim.}$$



Couple charged matter to (ext.) e.m. vector potential $\hat{A} = (A_\mu, A_{2n})$, $\mu = 0, \cdots, 2n-1$.

$A_{2n} =: \varphi$ ("axion")

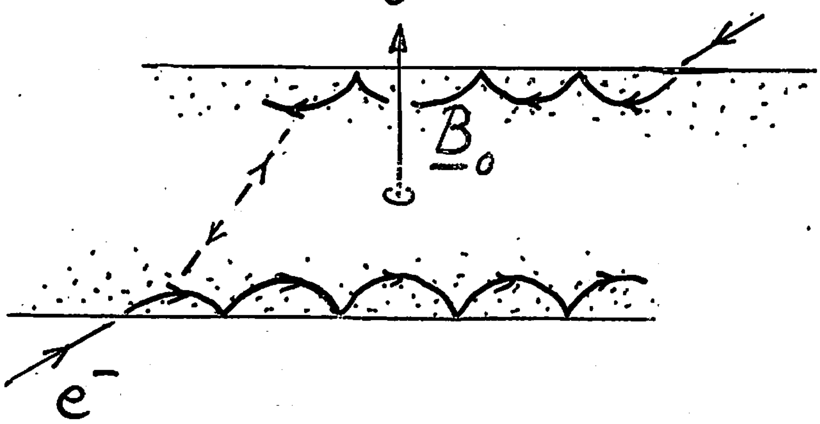
Interested in eff. action

$S^\Lambda_{eff}(\hat{A}) = S^\Lambda_{eff}(A, \varphi)$,

after matter fields (and other, massive gauge fields) have been integrated out.

Since $\partial\Lambda = \partial\Omega \times \mathbb{R} \neq \emptyset$ (bd. $2n-1$ branes), $\exists$ "massless" (charged) chiral surface modes ("diam. edge currents") $\Rightarrow$ anomalies (cancelling those of bulk eff. action) $\Rightarrow S^{\Lambda}_{eff}(\hat{A})$ contains bd. terms only dep. on $A := \hat{A}|_{\partial\Lambda}$.

Mass gen. via b-b tunnel.



tun. ampl.

$t\psi_L^* e^{iq\varphi}\psi_R + \cdots$

$\downarrow$

mass term

Some theor. considerations:

Couple e^- to pert., ext. vector potential $\hat{A}(\underline{x},t)$ $(=0, t \notin [-T,T])$,

$$H^{(0)} \mapsto H_t := H^{(0)} + \int_{\Omega} j_A^{\mu}(\underline{x}) \hat{A}_{\mu}(\underline{x},t)\, d\underline{x}$$

$\rightarrow$ Propagator $U_{\hat{A}}(t,s)$.

Partition function

$$Z_{\Omega}(\hat{A}) := \langle \varphi_0, U_{\hat{A}}(T,-T)\varphi_0 \rangle;$$

$$\Lambda := \Omega \times \mathbb{R}.$$

$$S_{eff}^{\Lambda}(\hat{A}) := -\ln Z_{\Omega}(\hat{A}).$$

Then

(1) $S_{eff}^{\Lambda}(\hat{A} + d\chi) = S_{eff}^{\Lambda}(\hat{A})$

(gauge invariance)

(2) $$\frac{\delta^n S^{\Lambda}_{eff}(\hat{A})}{\delta \hat{A}_{\mu_1}(x_1)\cdots\delta\hat{A}_{\mu_n}(x_n)} = \left\langle T\left[j^{\mu_1}(x_1)\cdots j^{\mu_n}(x_n)\right]\right\rangle^{conn.}_{\hat{A}}$$

(3) Bulk mass ("mobility gap") $> 0 \Rightarrow S^{\Lambda}_{eff}(\hat{A})$ local (exc. for bd. terms) $\rightarrow$ "gradient expansion": marginal term: CS + anomalous chir. bd. action

term of dim $4-(2n+1)$:

Maxwell term

(IR-irrelevant for $n=1$!)

(3) & (2) $\Rightarrow$ transport eqs.

Example: 2+1 D

$\rightarrow$ Hall's equ.

$$S_{eff}(\hat{A}) = \frac{\sigma_H}{2}\int \hat{A}\wedge d\hat{A}$$

$$+ \text{ bd. term} + \text{IR-irrel.}$$

$$\Leftrightarrow \quad \langle j^\mu(x)\rangle_{\hat{A}} = \sigma_H\, \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}(x)$$

or

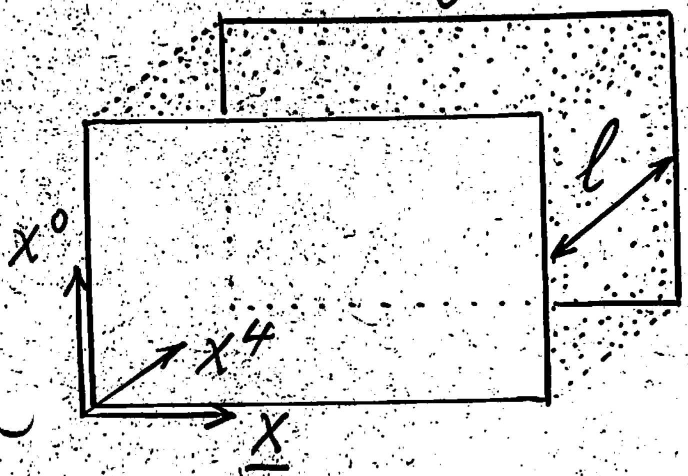
$$j^0(x) = \sigma_H B(x),$$

$$j^k(x) = \sigma_H \varepsilon^{kl} E_l(x)$$

Generalize to 4+1 D!

6. 4D/5D ANALOGON OF HALL EFFECT

"Space-time" Λ = slab of width $\ell \subset M^5$ (bd by 2 3-branes)



5D vector pot. $\hat{A}$, $\hat{A}_{\parallel}\big|_{\partial\Lambda} =: A$

Fill slab e.g. with massive four-component charged Dirac fermions.

$\longrightarrow$ Surface modes on $\partial\Lambda$ are massless, chiral, charged fermions

$$S_{eff}(\hat{A}) = \frac{1}{4\ell\alpha}\int_\Lambda d^5x\, \hat{F}_{\mu\nu}(x)\hat{F}^{\mu\nu}(x)$$

$$+ \frac{\sigma_H}{12}\int_\Lambda d^5x\, \varepsilon^{\mu\nu\rho\lambda\sigma}\hat{A}_\mu \hat{F}_{\nu\rho}\hat{F}_{\lambda\sigma}$$

$$+ \cdots + \Gamma_{\partial\Lambda}(A)$$

5D CS

chir. action of brane modes

$$A := \hat{A}_{\parallel}/_{\partial\Lambda}$$

$$\Rightarrow j^\mu = \langle j^\mu \rangle_{\hat{A}} = \frac{\delta S_{eff}(\hat{A})}{\delta \hat{A}_\mu}$$

$$= \frac{1}{4}\sigma_H \varepsilon^{\mu\nu\lambda\rho\sigma}\hat{F}_{\nu\lambda}\hat{F}_{\rho\sigma} + \cdots \quad (1)$$

5D anal. of Hall's law

$$j^{\mu}_{tot.} = j^{\mu}_{(bulk)} + j^{\mu}_{brane}$$

$$\partial_\mu j^{\mu}_{tot.} = 0$$

$$j^{\mu}_{brane} = j^{\mu}_{l}\,\delta_{\partial_- \Lambda} + j^{\mu}_{r}\,\delta_{\partial_+ \Lambda}$$

$$(1) \underset{\partial_\mu}{\Longrightarrow} \partial_\mu j^{\mu}_{l/r} = \mp\, \sigma_H\, \underline{E} \cdot \underline{B}$$

$$\Longrightarrow \quad \sigma_H = \sum_{\text{fermions}} \frac{q_i^3}{4\pi^2}$$

$$\left[\sigma_T = \sum \frac{q_i^2}{4\pi h}\left(\mu_r^i - \mu_l^i\right)\right]$$

4D chiral anomaly!

If $\hat{A}_\mu = A_\mu$, $\mu = 0,1,2,3$, and $\hat{A}_4 =: \varphi$ ("axion") are independent of x^4 then

$$S_{eff}(A;\varphi) = \frac{1}{4\alpha}\int_{\partial\Lambda} d^4x \{F^2(x) + 2(\nabla\varphi)^2(x)\} + \frac{\sigma_H}{12}\ell \int_{\partial\Lambda} \varphi (F_{\mu\nu}\tilde{F}^{\mu\nu})$$

P-Q

$$+ W_{\partial\Lambda}(A) + \cdots$$

eff. action of 4D massless fermions

$$\dot{\varphi} = \dot{\hat{A}}_4 = E_4$$

$$\Rightarrow \ell\dot{\varphi} = \int_0^\ell d\xi^4 E_4 = V = \mu_R - \mu_L$$

$$j^\mu = \delta S_{eff}(A;\varphi)/\delta A_\mu = \frac{1}{6}\sigma_H \ell\, \partial_\nu(\varphi \tilde{F}^{\nu\mu}) + \cdots \quad *$$

4D analogue of Hall eff.!

$$j^0 = \frac{\ell}{6}\sigma_H \, \underline{\nabla}(\varphi \underline{B}) + \cdots$$

$$\underline{j} = \frac{\ell}{6}\sigma_H \left\{ (\varphi \underline{B})^{\cdot} + \underline{\nabla} \wedge (\varphi \underline{E}) \right\} + \cdots$$

For $\varphi = \frac{3}{\ell q}(\mu_r - \mu_\ell)t$, <u>indep.</u> of $\underline{x}$, <u>no</u> magn. monopoles,

$$\underline{j} = \sigma_T \, \underline{B}$$

* in Maxwell's eqs. + eqs. of motion for φ:

$$\underline{\nabla} \cdot \underline{B} = j^0_M, \quad \underline{\nabla} \wedge \underline{E} + \dot{\underline{B}} = \underline{j}_M$$

$$\underline{\nabla} \cdot \underline{E} = \frac{\ell \sigma_H}{6} \left\{ (\underline{\nabla}\varphi) \cdot \underline{B} + \varphi j^0_M \right\}$$

$$\underline{\nabla} \wedge \underline{B} - \dot{\underline{E}} = \frac{\ell \sigma_H}{6} \left\{ \dot{\varphi} \underline{B} + \underline{\nabla}\varphi \wedge \underline{E} + \varphi \underline{j}_M \right\}$$

$$\Box\varphi = -\frac{\ell\,\sigma_H}{6}\,\underline{E}\cdot\underline{B} - \delta U(\varphi)/\delta\varphi$$

Non-linear PDE's

U is <u>periodic</u> in φ ...

For $j^{\mu}_{M} = 0$, special

solu is $\underline{E} = \underline{B} = 0$,

$\varphi = \varphi(t)$ solution of

$$\ddot{\varphi}(t) = -U'(\varphi(t))$$

<u>Pendulum!</u>

Linearization around:

Parametric resonance!

Unstable modes $\hat{\underline{E}}_{\underline{k}}$, $\hat{\underline{B}}_{\underline{k}}$,

$$\underline{E}\cdot\underline{B} \neq 0!$$

A simple special case:

$$\underline{j} = \sigma_T \underline{B} + \sigma_\Omega \underline{E}$$

$$\sigma_T = \sum_i \frac{q_i^2}{4\pi h}(\mu_r^i - \mu_\ell^i)$$

$\sigma_\Omega \gg \sigma_T > 0$: primordial plasma

Maxwell's eqs.

$$\underline{\nabla}\cdot\underline{B} = 0, \quad \underline{\nabla}\cdot\underline{E} = 0.$$

$\Rightarrow$ $\underline{E}, \underline{B}$ transv. pol.

$$\underline{\nabla}\wedge\underline{E} + \dot{\underline{B}} = 0, \quad \underline{\nabla}\wedge\underline{B} - \dot{\underline{E}} = \sigma_T\underline{B} + \sigma_\Omega\underline{E}$$

Fix wave vector $\underline{k} = k\underline{e}_3$, $k>0$; $\underline{X}^T$: comp. of $\underline{X} \perp \underline{k}$

Then

$$\begin{pmatrix} \underline{\dot{E}}^T \\ \underline{\dot{B}}^T \end{pmatrix} = K(k) \begin{pmatrix} \underline{E}^T \\ \underline{B}^T \end{pmatrix}, \quad \text{with}$$

$$K(k) = \left(\begin{array}{cc:cc} -\sigma_\Omega & 0 & -\sigma_T & -ik \\ 0 & -\sigma_\Omega & ik & -\sigma_T \\ \hdashline 0 & ik & 0 & 0 \\ -ik & 0 & 0 & 0 \end{array}\right)$$

Eigenvalues of $K(k)$: $i\omega_\alpha(k)$, $\alpha = 1, \dots, 4$ (circ. frequ. of normal modes).

If $i\omega_{\alpha_0}(k) > 0 \Rightarrow$ expon. growing normal mode!

$$i\omega(k) = \frac{-\sigma_\Omega \pm \sqrt{\sigma_\Omega^2 - 4k(k \pm \sigma_T)}}{2}$$

(hand-made calculation)

$\Longrightarrow$ For $0 < k < \sigma_T$, $\exists!$ one positive solution,

$$i\omega_{\alpha_0}(k) \approx \frac{k(\sigma_T - k)}{\sigma_\Omega}$$

(1) Sources for large initial axion field φ:

- Gravity
  $\varphi \, \mathrm{tr}(R \wedge R)$ – term
  ("wobbling" geometry)
- Phase transition
  $$U = 0 \xrightarrow[T \searrow]{} U \neq 0$$

- - - - - - - -

(2) Connection with R-S

$$y := x^4$$

$$ds^2 = e^{2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2$$

eq. for warp factor $e^{2\sigma}$!

(mass) hierarchy,
cosmological
constant, ?

origin of axion (?)

Can render fifth di-
mension discrete
(à la Connes-Lott et al.)
and still get all
of the above;
(A.H.C & J.F., Lizzi et al.)

Applications.

- Growth of seed magn. fields in early universe

- Growth of textures in superfluid ^3He-A (Volovik)

THE END FOR

SUMMER OF 2004